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DISTRIBUTION OF DENSITY IN AN ION-EXOSPHERE OF A NON-ROTATING PLANET

by

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ABSTRACT

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A model of an ion exosphere is set up and an expression is derived for the variation of density with altitude. The ions are considered to be under the control of superimposed gravitational and static, centered-dipole magnetic fields and to have mirror points in the barosphere. Collisions are ignored and the particles are assumed to be trapped in the dipole field.

By use of the first adiabatic invariant and Liouville's theorem, the density is found to fall off faster with height in the exo-magnetosphere than in either a neutral exosphere or an ionized barosphere. A justification of the use of reduced effective gravity for the scale height of the ions is given in the appendix.

Author

INTRODUCTION

The base of a true ion-exosphere, i.e. a region, in which ionic mean free paths are of the same order of magnitude as the path length along a line of force, will be much higher than that of the neutral exosphere, since the Coulomb cross-section is much greater than the gas-kinetic cross-section. Such an exosphere will have its base in a magnetospheric region in which the ion density is of the order of a few hundred per cubic centimeter. The complete physical problem is one of immense complexity, even when the neutral particles are ignored, involving, inter alia, various mechanisms of ion creation and loss, interactions with gravitational, magnetic and, as yet unspecified electric fields, diffusion processes of various types and even interplanetary plasma and the solar wind.

In order to simplify the problem, we shall consider only the effects of the superimposed gravitational and static, centered-dipole magnetic fields upon ions, the mirror points of which are below an imaginary interface, the baropause, separating the lower region, the barosphere, in which charged particles are in hydrostatic equilibrium from the upper region, the exosphere, in which collisions can be ignored. There are no escape orbits, for we assume that all of the charged particles are trapped by the magnetic field. Therefore, we must take limiting exospheric case of no collisions whatsoever, for even very rare collisons would, in the case of trapped particles, lead eventually to the establishment of hydrostatic equilibrium. Under this assumption, there will be no particles with mirror points in the exosphere itself. We shall also assume that at the baropause, the velocity distribution is Maxwellian, with a common temperature for both the ions and the electrons. It is shown in the appendix that, from the existence of such a common temperature, it follows that the effect of the electron s on the motion of the ions is to reduce their gravitational acceleration by a factor of $(1 + Z)$, where Z is the degree of ionization.

The problem of computing the density distribution for the isolated neutral exosphere has been treated by Öpik and Singer (1959; 1961), who derived an expression for the density distribution as a function of altitude. However, the analogous problem for an ionized exosphere involves consideration of the effect of the magnetic field, as well as the gravitational field. Johnson (1959; 1960) has discussed the distribution of protons in the region above 800 km, using a barometric neutral distribution, modified by an inverse cube attenuation. In Johnson's model, the particles are envisaged as spreading apart as they go upward, and crowding together as they go downward. In our model, this conduct is characteristic only of those particles, whose speed is below a certain critical value; the faster particles on the ohter hand, tend to exhibit the opposite type of behavior, i.e. their pitch angles are crowded into a steadily narrowing cone on the way up and spread out on the way down. As a consequence, we derive a different altitude dependence of density.

MOTION OF AN ION

The scale of the gravitational field is large compared to the radius of curvature of ion's trajectory in the magnetic field. Therefore, one can apply Alfven's perturbation method and the guiding center approximation, conserving the magnetic moment, or first invariant, defined as follows,

$$\mu = mv^2 \sin^2\alpha \,/2B = \text{const.} \tag{1}$$

where m is the ion mass, v the magnitude of its velocity, α its pitch angle and B the magnitude of the magnetic field.

We also make use of the fact that a static magnetic field does no work on a charged particle moving in it, which enables us to write down, immediately, the law of conservation of energy:

$$v_{\parallel}^2 + v_{\perp}^2 = v_{\parallel o}^2 + v_{\perp o}^2 + 2\gamma(1-R/r)/R \qquad (2)$$

Here, and in all the subsequent discussions, the subscript zero attached to a quantity refers to the value of that quantity at the baropause. The subscripts $_{\parallel}$ and $_{\perp}$ refer to directions, parallel and perpendicular respectively, relative to the local direction of the line of force, along which the guiding center of the particle is moving, r is the geocentric distance of the particle, R is the radius of the baropause and γ is the product of the gravitational constant and the mass of the earth. The variations of the quantities are taken along the line of force. Since the gravitational force is conservative, it can be integrated along the line of force to give the difference in gravitational potential between any two points on the line of force.

Using equation (1), equation (2) can be written in the form:

$$v_{\parallel}^2 - v_{\parallel o}^2 = -2\mu\ (B-B_o)/m - 2\gamma(1 - R/r)/R \qquad (3)$$

We now define dimensionless variables, in terms of which the remainder of the calculations will be carried out.

$$v_{\infty}^2 = 2\gamma/R \quad \underline{U} = \underline{v}/v_{\infty} \ ; \ y = R/r; \quad \eta = B/B.$$

In terms of these variables and the explicit form of μ, equation (3) now has the form

$$u_{\parallel}^2 = u_o^2 \, (1 - \eta \sin^2 \alpha_o) - (\, (1 - y) \tag{4}$$

and the law of conservation of energy becomes:

$$u^2 = u_o^2 - (1 - y) \tag{5}$$

The ratio of the magnetic fields in these units is given by:

$$\eta = y^3 \, (4-3y_e/y)^{\frac{1}{2}}/(4-3y_e) \tag{6}$$

where y_e is the value of y at the geomagnetic equator. If R and y_e are given, the line of force is uniquely specified by means of its equation:

$$y = y_e/\cos^2 \lambda \tag{7}$$

where λ is the geomagnetic latitude. The parameter y_e differs from the usual L in that it is measured in baropause radii instead of the usual earth radii.

THE DISTRIBUTION OF IONS

For non-relativistic particles, Liouville's theorem,

$$D = dN/d^3 \, \underline{x} \, d^3 \, \underline{p} = \text{const.} \tag{8}$$

tells us that the differential directional intensity, j, is proportional to u^3, and, having been isotropic at the base, it will remain isotropic everywhere, in the sense that it remains uniform inside the allowed cone. The ratio between the omnidirectional differential intensity J (y) and J_o is given by:

$$J/J_o = (u/u_o)^3 (\Omega/\Omega_o) \tag{9}$$

where Ω is the solid angle subtended by the allowed cone.

$$\Omega/\Omega_o = \int d\Omega \Big/ \int d\Omega_o = \int d(\cos\alpha) \Big/ \int d(\cos\alpha_o) \tag{10}$$

A particle will reach a level y, if, and only if, it has sufficient kinetic energy to overcome the gravitational forces, or in terms of the parallel velocity, $u_{11}(y) \geq 0$, or using equation (4):

$$u_o \geq \left\{(1 - y)/\ (1 - \eta\sin^2\alpha_o)\right\}^{\frac{1}{2}} \tag{11}$$

The lowest possible initial velocity is that of a particle having only parallel motion, $\alpha_o = 0$, giving:

$$\min\left\{u_o\right\} = (1 - y)^{\frac{1}{2}} \tag{12}$$

Contributions to the intensity at y, for a given initial velocity, will come only from a cone of aperture α_m, where α_m is the initial pitch angle corresponding to a pitch angle $\alpha = \pi/2$ at y. The cone of aperture $\alpha_m(u_o, y)$ at the barosphere opens up into a complete hemisphere at y.

On the other hand, there exist particles which originally have no parallel velocity ($\alpha_o = \pi/2$), which nonetheless arive at y, having mirrored up from the baropause, through the effect of the magnetic field. Such particles must have initial velocities which satisfy the inequality:

$$u_o \geq \left\{(1 - y)/(1 - \eta)\right\}^{\frac{1}{2}} \tag{13}$$

At the baropause, these particles fill the entire hemisphere. However, at y, their pitch angles have been reduced in accordance with equation (1), and they are now crowded into a cone of aperture α_m', given by means of equations (1) and (5), as:

$$\alpha'_m(u_o, y) = \text{arc sin } \left\{ \eta u_o^2/(u_o^2 - (1-y)) \right\}^{\frac{1}{2}} \tag{14}$$

This crowding of the fast particles into a cone is in contradiction to the picture visualized by Johnson (1959).

We must separate the ions into two classes, according to their initial velocities, and associate a different solid angle ratio with each class:

$$\text{Class A:} \quad (1-y)^{\frac{1}{2}} < u_o < \left\{ (1-y)/(1-\eta) \right\}^{\frac{1}{2}}$$

$$\text{Class B:} \quad \left\{ (1-y)/(1-\eta) \right\}^{\frac{1}{2}} < u_o < \infty$$

For class A, the solid angle ratio is unity, for, as mentioned above, the allowed trajectories fill the entire hemisphere both at $y = 1$ and at any value of y. However, for class B, the solid angle at $y < 1$ is reduced from the hemisphere subtended at the base. Using equation 10 , the solid angle ratios are:

$$(\Omega/\Omega_o)_A = 1 \tag{15}$$

$$(\Omega/\Omega_o)_B = 1 - \cos \alpha'_m = 1 - \left\{ (u_o^2(1-\eta) - (1-y)) \,|\, (u_o^2 - (1-y)) \right\}^{\frac{1}{2}}$$

where the explicit form of $\cos \alpha'_m$ has been obtained from equation (14).

The ratio of the differential densities is:

$$n/n_o = (u/u_o)^2 \quad (\Omega/\Omega_o) \tag{16}$$

The total relative ion density, normalized to unity at the baropause, is obtained by integrating the above over the Maxwellian distribution prevailing at the baropause.

$$\frac{N(y,\ y_e)}{N(1,\ y_e)} = \frac{4E^{3/2}}{\sqrt{\pi}} \left\{ \int_{(1-y)^{\frac{1}{2}}}^{\left(\frac{1-y}{1-\eta}\right)^{\frac{1}{2}}} du_o\, u_o^2 \left(\frac{u}{u_o}\right)^2 \left(\frac{du}{du_o}\right) \left(\frac{\Omega}{\Omega_o}\right)_A \exp(-Eu_o^2) \right.$$
$$\left. + \int_{\left(\frac{1-y}{1-\eta}\right)^{\frac{1}{2}}}^{\infty} du_o\, u_o^2 \left(\frac{u}{u_o}\right)^2 \left(\frac{du}{du_o}\right) \left(\frac{\Omega}{\Omega_o}\right)_B \exp(-Eu_o^2) \right\} \tag{17}$$

where the escape parameter at the baropause is:

$$E = m\, v_\infty^2 / 2\, k\, T \tag{18}$$

We insert the expressions for $\left(\frac{\Omega}{\Omega_o}\right)$ A, B from equation (15) and for u and its derivative from equation (5), obtaining:

$$\frac{N(y,\ y_e)}{N(1,\ y_e)} = \frac{4E^{3/2}}{\sqrt{\pi}} \left\{ \int_{(1-y)^{\frac{1}{2}}}^{\infty} du_o\, u_o \left(u_o^2 - (1-y)\right)^{\frac{1}{2}} \exp(-Eu_o^2) \right.$$
$$\left. - \int_{\left(\frac{1-y}{1-\eta}\right)^{\frac{1}{2}}}^{\infty} du_o\, u_o \left(u_o^2 (1-\eta) - (1-y)\right)^{\frac{1}{2}} \exp(-Eu_o^2) \right\} \tag{19}$$

Both integrals can be calculated in closed form, giving, after some manipulation:

$$N(y,\ y_e)/N(1,\ y_e) = \exp(-E(1-y)\ )\ \left(1-(1-\eta)^{\frac{1}{2}}\ \exp(-E\eta(1-y)/(1-\eta))\right) \tag{20}$$

Using the explicit form of η given in equation (6), one may compute values of the relative density in various ways. In graph 1, the variations in density of hydrogen ions along a radius are presented for radii passing through three different latitudes, assuming a temperature of 1500°K at the baropause. In graph 2 the variation along the line of force crossing the geomagnetic equator at 2 barosphere radii is given. Graphs 3, 4 and 5 show the zonal variation of density for 1500° and 2000° hydrogen at geocentric distances of 2, 5, and 10 barosphere radii. In all cases the baropause is at 1.1 earth radii.

The density in an ion exosphere is reduced below that of a barosphere by a factor which depends upon the variation of the magnetic field upon the geomagnetic latitude and the height of the point in question. It is of interest to note the comparison in graph 1 between our density curves and that of Öpik and Singer from their 1959 paper (op. cit.). In both cases the exosphere density lapse is greater than that of the corresponding barometric distribution. However, at low and middle latitudes, the decrease in scale height is greatest in the ionized exosphere, as might be expected on purely intuitive grounds. The barometric proton scale height without the magnetic field is twice that of neutral hydrogen because of the effective gravity reduction, while in the exomagnetosphere, the lack of energy partition and the constraint of motion along magnetic field lines function together to inhibit the upward motion of the ions. The last effect diminishes with increasing latitude, as indicated by graphs 3, 4, and 5, for as one goes poleward, the lines of force tend to become more nearly vertical.

APPENDIX

JUSTIFICATION OF THE USE OF REDUCED GRAVITY IN THE ION EXOSPHERE PROBLEM

Assuming that the ions and electrons have a common temperature, gross quasineutrality of the plasma can only be maintained if the following approximate relation holds

$$m_e g_e \approx m_i g_i \tag{1}$$

where the subscripts e and i refer to the electrons and ions respectively, m is mass and g the effective gravitational acceleration. As is well known, equation (1) leads to the conclusion that there must be an electric field which reduces the effective gravity of the ion below the normal gravitational acceleration be a factor of $(1 + Z)$, where Z is the degree of ionization.

It will now be shown that if the deviation from charge neutrality exceeds a fraction of a percent of the total number density, the electrostatic energy generated will exceed the thermal energy of the plasma.

Assume that above a certain level R, there is a space charge caused by deviations of the plasma from neutrality; these deviations are a result of scale height inequality.

$$H_i \ll H_e \tag{2}$$

The electrostatic potential created by this charge density will be:

$$\phi\ (r) = \int |\underline{r}-\underline{r}'|^{-1}\ r'^2\ dr'\ \rho(r')d\Omega \tag{3}$$

We shall assume the following form for the function $\rho(r)$

$$\rho(r) = \begin{cases} \rho_o \exp(-(r-R)/H) & r > R \\ 0 & r < R \end{cases} \tag{4}$$

$\rho_o = \epsilon\Delta N$, where ϵ is the electronic charge and ΔN the deviation from neutrality.

This is a two-dimensional problem. We integrate over all space outside a large sphere of radius R and find the potential drop over one scale height $H \ll R$. The function in (3) can be expanded in the usual Legendre polynominals.

$$|\underline{r} - \underline{r}'|^{-1} = \begin{cases} r^{-1} \Sigma(r'/r)^{n} \, Pn(\cos\theta); \; r' < r \\ \Sigma(r/r')^{n+1} \, Pn(\cos\theta); \; r' > r \end{cases} \tag{5}$$

$$\rho(r) = 2\pi\rho_o \left\{ \int d(\cos\theta) \Sigma \, Pn(\cos\theta) \;\; [r^{-(n+2)} \int_R^r dr' r'^{n+2} \exp(-(r'-R)/H) + r^n \int_r^\infty dr' r'^{n+1} \exp(-(r'-R)/H)] \right. \tag{6}$$

Integrating over r' and $\cos\theta$ and using the orthonormality of the Legendre polynomials, we obtain, after defining:

$$X = (r-R)/H$$

$$\emptyset(r) = 4\pi\rho_o \;\; r^{-1} \, [H(R^2 - r^2 e^{-X}) + 2H^2(R - re^{-X}) + 2H^3(1-e^{-X})] + H(H+r)e^{-X} \tag{7}$$

Now consider a volume consisting of a column of unit cross section and height $H \ll R$ and r, and apply the condition that the electrostatic energy stored in the layer between R and R+H be no greater than the thermal energy of the plasma of number density N, contained in the column. This will give us an upper bound for $\Delta N/N$. Under the approximation $(R/H)^2 \gg (R/H) \gg 1$, we have:

$$E_{elec.} = \epsilon(\emptyset(R+H) - \emptyset(R)) = -4\pi\epsilon^2 \Delta N H^2 \qquad (8)$$

$$E_{thermal} = NHkT > |E_{elec.}| \qquad 9$$

giving:

$$\Delta N/N < kT/4\pi\epsilon^2 H \qquad 10$$

For H = 100 km and T = 1500° K, we have

$$\Delta N/N < 7.1 \times 10^{-3} \qquad 11$$

For the same temperature and a hydrogen ion density of 5.10^3 cm^{-3} typical of a geocentric distance of 1.4 earth radii (Singer and Lenchek, 1962), the Debye length may be estimated. Taking the limiting case of equality in equation (11), we find for the Debye length

$$h = 6.90\ (T/\Delta N)^{\frac{1}{2}} = 120\ \text{cm.}$$

i.e. the Debye length, which is the distance over which the plasma can deviate appreciably from neutrality is of the order of a meter, considerably smaller than the scale height or any of the other characteristic dimensions involved in the problem.

Equation (11) is, at best, an order of magnitude estimation, for it is doubtful that the electrons obey such a barometric density lapse law in the exosphere. However, it is clear that even if the density lapse is steeper than the barometric lapse, the conditions on the relative deviation will become more stringent and our point is even more strongly established. The assumption that the deviation obeys the electrons' distribution law merely strengthens our conclusion, for the ion scale height, if assumed to be much smaller than that of the electrons, will affect the result only slightly. What we have shown is, in fact, that the ion and electron scale heights must be very close to equality, for if not, electric fields will develop which will tend to restore the gross neutrality of the plasma. It is precisely such a field which determines the effective gravity of the ions. It should be noted that this result does not require that any particles escape, and, therefore, the presence or absence of a static magnetic field which traps the particles, but does no work upon them, does not affect the result in any way.

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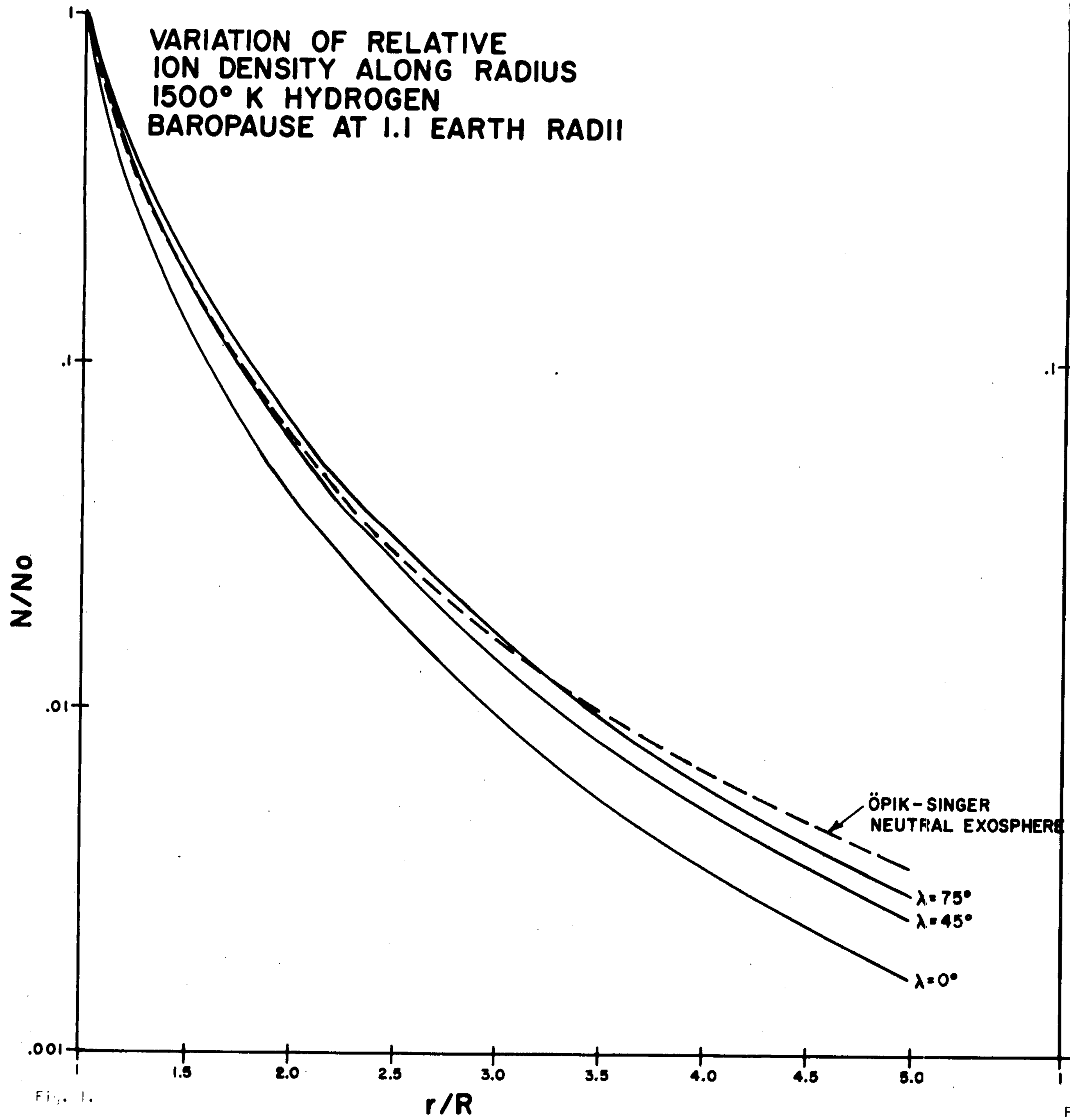

VARIATION OF RELATIVE
ION DENSITY ALONG RADIUS
1500° K HYDROGEN
BAROPAUSE AT 1.1 EARTH RADII
N/No
1
.1
.01
.001
1
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
r/R
ÖPIK-SINGER
NEUTRAL EXOSPHERE
λ=75°
λ=45°
λ=0°

Fig. 1.

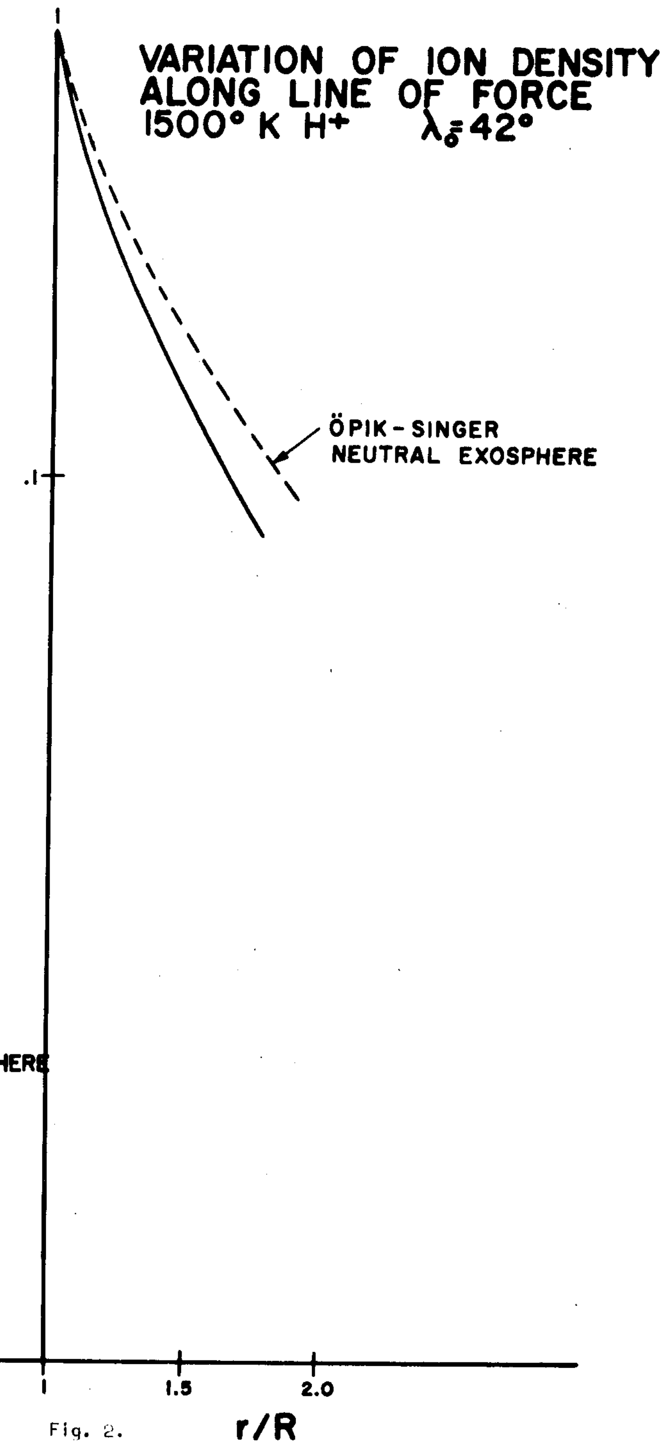

VARIATION OF ION DENSITY
ALONG LINE OF FORCE
1500° K H+ λ₀=42°
1
.1
1
1.5
2.0
r/R
ÖPIK-SINGER
NEUTRAL EXOSPHERE

Fig. 2.

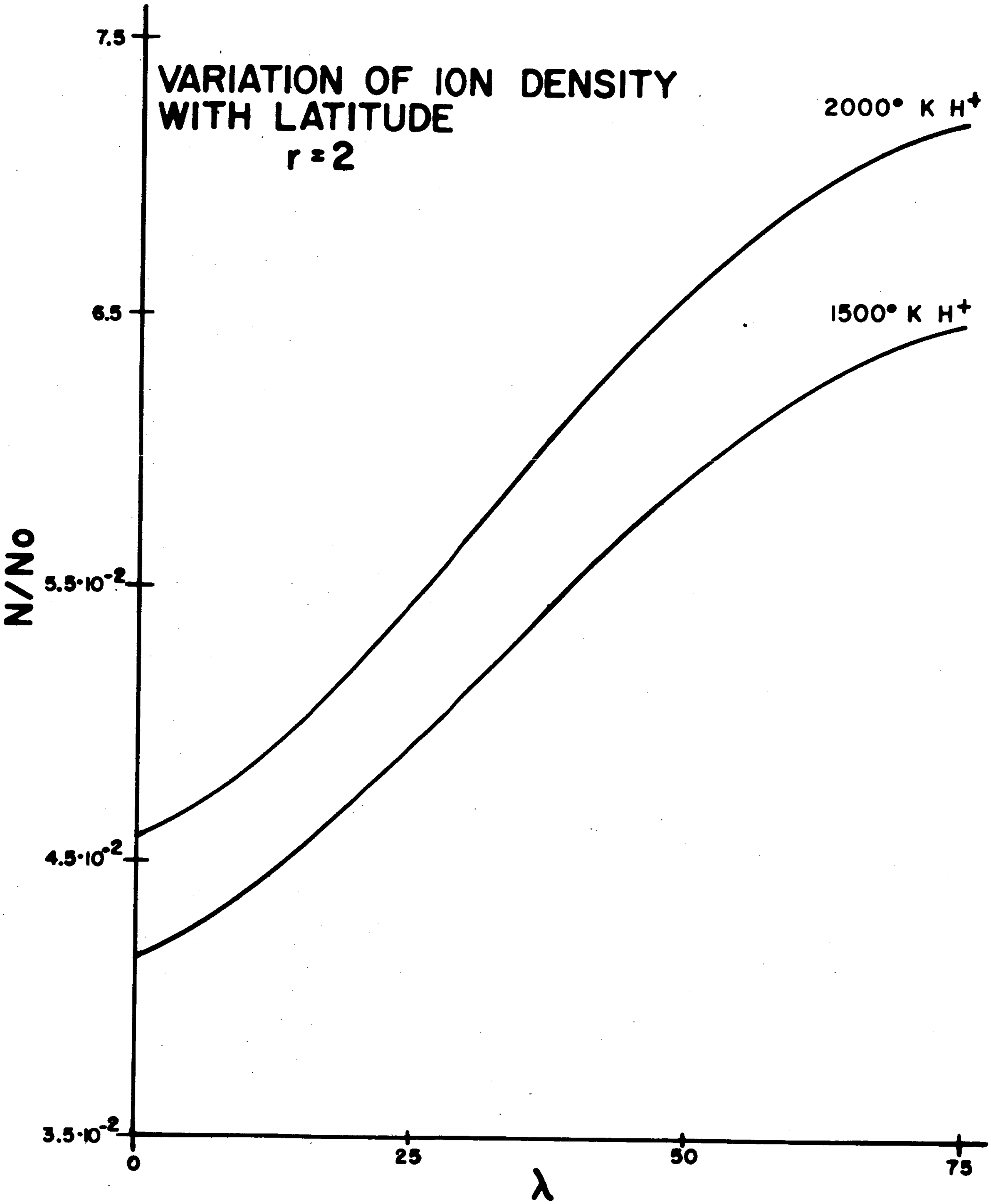

VARIATION OF ION DENSITY
WITH LATITUDE
r = 2
2000° K H+
1500° K H+
N/No
7.5
6.5
5.5·10-2
4.5·10-2
3.5·10-2
0
25
50
75
λ

Fig. 3.

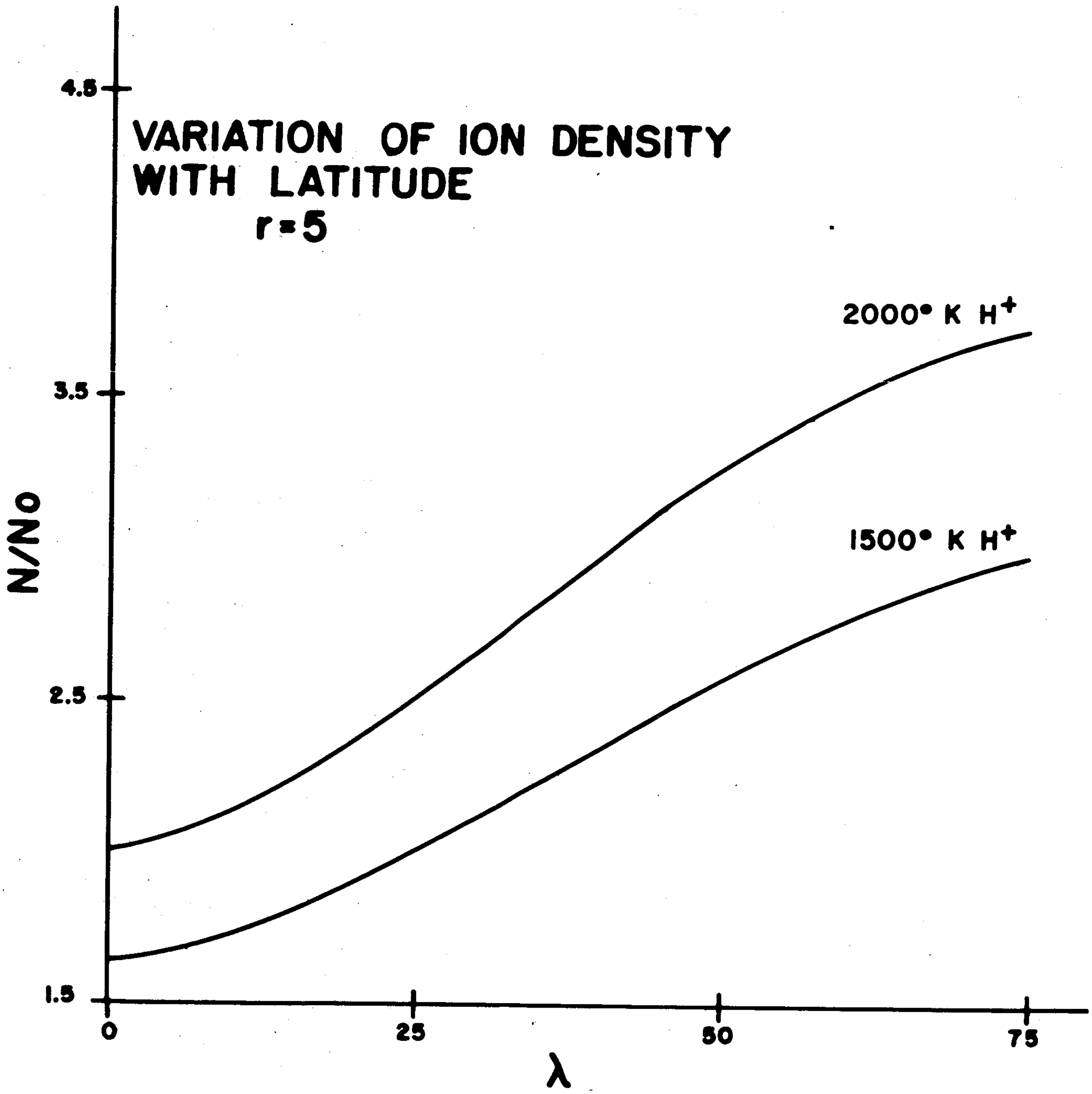
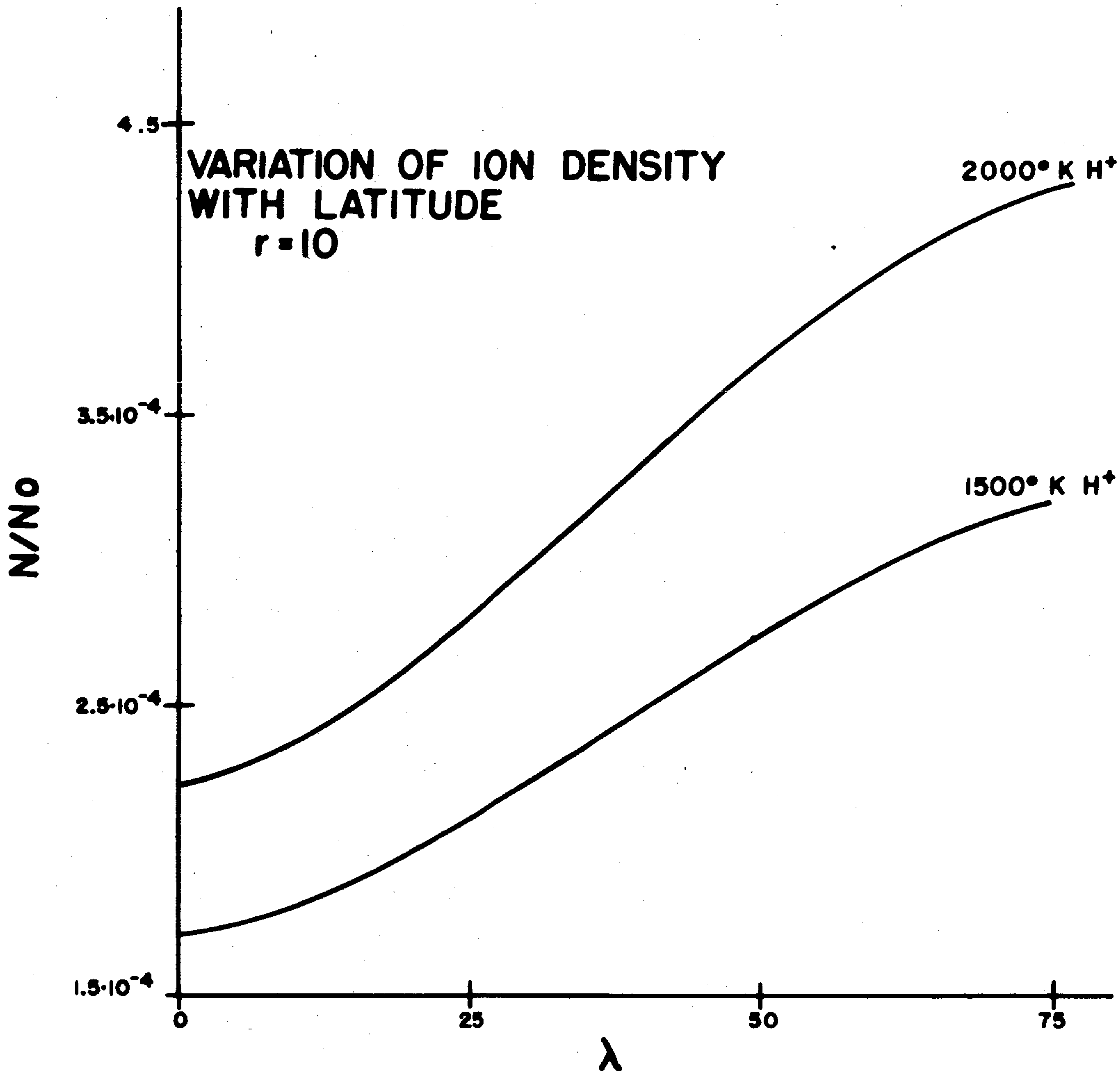

VARIATION OF ION DENSITY WITH LATITUDE
r=5
2000° K H+
1500° K H+
N/No
4.5
3.5
2.5
1.5
0
25
50
75
λ

Fig. 4.



VARIATION OF ION DENSITY WITH LATITUDE
r = 10
2000° K H+
1500° K H+
4.5
3.5·10^-4
2.5·10^-4
1.5·10^-4
N/No
0
25
50
75
λ

Fig. 5.